# Momentum Map and Action-Angle Variables
# for Nambu Dynamics

Adnan Teğmen

Department of Physics, University of Ankara, Faculty of Sciences, 06100 Ankara, TURKEY
*tegmen@science.ankara.edu.tr*

**Abstract**

Momentum map is a reduction procedure that reduces the dimension of a Hamiltonian system to the lower ones. It is shown that behavior of the action-angle variables under the momentum map generates the new action-angle variables for the reduced system considered as a Nambu structure. The symmetrical top is given as an illustration.

*arXiv:math-ph/0508058v1  29 Aug 2005*

## 1    Introduction

It is well known that the time evolution of a Hamiltonian dynamical system with the Hamiltonian $H$ and $n$ degrees of freedom is governed by the equations

$$\dot{q}_i = \frac{\partial(q_i, H)}{\partial(q_i, p_i)}, \qquad \dot{p}_i = \frac{\partial(p_i, H)}{\partial(q_i, p_i)}, \qquad i = 1, ..., n, \tag{1}$$

where $q_i$ and $p_i$ are the conventional phase-space coordinates, *i.e.*, generalized coordinates and their conjugate momenta, respectively. It is obvious that dimension of the phase space of the standard Hamiltonian system is even. But in 1973 Yoichiro Nambu introduced a generalization of Hamiltonian mechanics for arbitrary-dimensional phase spaces [1]. His formalism is based on an $N$-tuple of phase space variables and $N-1$ Hamiltonians considered as integrals of motion. He replaced the usual Poisson bracket that is a binary operation by his own bracket as an $N$-ary operation. In this manner, this generalization allows us to define odd-dimensional phase spaces besides the even ones. Indeed, if we try to determine (1), for instance, for a phase space described by the coordinates $x_1, x_2, x_3$, we should write

$$\dot{x}_i = \frac{\partial(x_i, H_1, H_2)}{\partial(x_1, x_2, x_3)}, \qquad i = 1, 2, 3. \tag{2}$$

Nambu focused his formalism on the three dimensional phase spaces giving Euler equations for the free rigid body as an application. After Nambu's proposal some authors analyzed this formalism. For example, in Ref. [2] Nambu dynamics is treated as six-dimensional degenerate Hamiltonian system with three constraints. In Ref. [3] it is shown that Nambu formalism is an embedding of three-dimensional phase space into the four-dimensional one.

In the proceeding years there were attempts to embody the Nambu formalism in a geometric framework. Ref. [4] treats the formalism on a mathematical basis and also contains a generalization of the Jacobi's identity, so-called fundamental identity.

The aim of this work is to construct the variables in the Nambu mechanics corresponding to the action-angle variables of Hamilton mechanics. To do so, rather than to guess intuitively, invariance of momentum map under canonical transformations is used. First we give a brief summary about symplectic structures that is a geometric description of Hamiltonian systems and their action-angle variables.

The phase space of a dynamical system with $n$-degrees of freedom is the cotangent bundle $T^*Q = M$ of the configuration space $Q$ that is the set of all possible spatial positions of objects in the system. $M$ has a natural symplectic structure given by a closed, nondegenerate differential 2-form

$$\omega^{(2)} = dq_i \wedge dp_i, \qquad i = 1, ..., n, \tag{3}$$

where $\wedge$ is the exterior product. (Throughout the article the Einstein summation rule is used for repeated indices). The dynamics of a mechanical system is specified by a differentiable real-valued function $H$, the Hamiltonian, defined on M, *i.e.*, $H : M \to R$. The time evolution of the system is given by the integral curve $q_i(t)$ of the corresponding Hamiltonian vector field $X_H$ defined by

$$X_H = {}^\sharp dH. \tag{4}$$

Here the linear mapping $\sharp : T^*M \to TM$, for $2n$-dimensional phase space with canonical coordinates $(q_1, p_1, ..., q_n, p_n)$, is defined by

$${}^\sharp dq_i = \partial_{p_i} , \quad {}^\sharp dp_i = -\partial_{q_i} , \tag{5}$$

where TM stands for the tangent bundle of M. Then the Hamiltonian vector field $X_H$ is explicitly

$$X_H = \partial_{q_i}H\partial_{p_i} - \partial_{p_i}H\partial_{q_i} . \tag{6}$$

The Hamilton equations of motion are therefore obtainable easily by

$$\dot{x} = -{}^\sharp dH(x), \tag{7}$$

where $x \in \{q_1, p_1, ..., q_n, p_n\}$.

The phase space arising in Hamiltonian mechanics has an additional structure called Poisson bracket. If $f$ and $g$ are smooth, real valued functions on $M$, the Poisson bracket $\{ , \}_P$ in canonical coordinates is defined to be

$$\{f, g\}_P = \partial_{q_i}f\partial_{p_i}g - \partial_{p_i}f\partial_{q_i}g. \tag{8}$$

The bracket operation can be defined entirely in terms of the symplectic form without stating any particular coordinate system,

$$\omega^{(2)}({}^\sharp df, {}^\sharp dH) = \{f, H\}_P. \tag{9}$$

Here the left hand side of (9) is the determinant defining the pairing of a 2-form and the tangent bundle vector fields;

$$\begin{vmatrix} \langle {}^\sharp df, dq_i\rangle & \langle {}^\sharp df, dp_i\rangle \\ \langle {}^\sharp dH, dq_i\rangle & \langle {}^\sharp dH, dp_i\rangle \end{vmatrix}, \tag{10}$$

where $\langle , \rangle$ is pairing operator defined by $\langle \partial_{x_i}, dx_j\rangle = \delta_{ij}$. Then the Hamiltonian equations of motion read

$$\dot{q}_i = \omega^{(2)}({}^\sharp dq_i, {}^\sharp dH) , \quad \dot{p}_i = \omega^{(2)}({}^\sharp dp_i, {}^\sharp dH). \tag{11}$$

It is well known that the canonical transformations are the mappings $T : M \to M$ which preserve the form of Hamiltonian equations of motion. In the symplectic context, canonical transformations are the diffeomorphisms which preserve the symplectic structure and are sometimes called symplectomorphisms. The coordinate set of the action-angle variables $(\mathcal{I}_i, \theta_i)$ is a special kind of canonical transformations, such that the angle variables $\theta_i$ appear as the cyclic coordinates in the new system and therefore the action variables $\mathcal{I}_i$ are constants of motion:

$$\dot{\theta}_i = \partial_{\mathcal{I}_i} K, \qquad \dot{\mathcal{I}}_i = -\partial_{\theta_i} K = 0, \tag{12}$$

where $K$ is the transformed Hamiltonian. According to the argument stated above, the action-angle variables argue the relation of invariance,

$$d\mathcal{I}_i \wedge d\theta_i = dq_i \wedge dp_i. \tag{13}$$

Shape invariance of the Hamiltonian equations under the transformation $(q_i, p_i) \to (\mathcal{I}_i, \theta_i)$ induces the invariance of the closed areas $A_i$ projected onto the $(q_i, p_i)$ planes. Thus under this circumstance the action variables are defined in terms of these areas:

$$\mathcal{I}_i = \frac{1}{2\pi} \oint p_i dq_i = \frac{1}{2\pi} A_i. \tag{14}$$

On the other hand the quantities $\dot{\theta}_i = \nu(\mathcal{I}_i)$ correspond to the sweeping frequencies of the areas $A_i$.

## 2  Nambu Structure and Momentum Map

Nambu structure is a generalization of the Hamiltonian structure, proposed by Y. Nambu in 1973 [1]. In Nambu's formalism the equations of motion for $N$-dimensional phase space with coordinates $x_1, \ldots, x_n$, are governed by $N-1$ Hamiltonians $H_1, \ldots, H_{N-1}$ and a closed, nondegenerate $N$-form $\omega^{(N)} = dx_1 \wedge \cdots \wedge dx_N$ (volume form on Nambu phase space $M_N$) :

$$\dot{x}_i = \omega^{(N)}(\widetilde{dx_i}, \widetilde{dH_1}, \ldots, \widetilde{dH_{N-1}}) \ , \quad i \in \{1, \ldots, N\}, \tag{15}$$

here we adopt the linear map $\sim$ defined by

$$\widetilde{dx_i} = \partial_{x_i}, \tag{16}$$

so that $\widetilde{df} = \partial_{x_i} f \partial_{x_i}$. (15) defines the Nambu bracket $\{., \ldots, .\}_N$

$$\omega^{(N)}(\widetilde{dx_i}, \widetilde{dH_1}, \ldots, \widetilde{dH_{N-1}}) = \{x_i, H_1, \ldots, H_{N-1}\}_N, \tag{17}$$

which is given explicitly by the Jacobian

$$\frac{\partial(x_i, H_1, \ldots, H_{N-1})}{\partial(x_1, \ldots x_N)} = \{x_i, H_1, \ldots, H_{N-1}\}_N, \tag{18}$$

that is the generalization of the usual Poisson bracket just as stated originally by Nambu. It is remarkable to point out that the systems with $n$-degrees of freedom $(n \geq 2)$, *i.e.*, even dimensional phase space $(N = 2n)$ with the canonical coordinates $(q_1, p_1, \ldots, q_n, p_n)$, can be treated in two ways.

First, it can be considered as a Hamiltonian system with the usual Poisson bracket. Second, it can be interpreted as a Nambu system with the bracket

$$\dot{f} = \frac{\partial(f, H_1, \ldots, H_{N-1})}{\partial(q_1, p_1, \ldots, q_n, p_n)}. \tag{19}$$

In this case there appears an inevitable normalization function that is also a constant of motion in front of the right hand side of (18) [5, 6, 7]. In order to get the correct equations of motion the Nambu bracket must then be normalized properly. Since this approach does not destroy the fundamental ideas within this paper, all Nambu brackets will be considered as normalized.

On the other hand when considering the odd dimensions the phase-space coordinates are not the canonical ones in the Poisson sense. The systems with the non-canonical coordinates can be defined as the substructure of a standard Hamiltonian system [8]. A general feature for Hamiltonian systems with symmetry, for instance integrable or superintegrable, is that it is possible to reduce the dimension of the system to the lower ones. This reduction procedure is achieved by appealing to non-canonical coordinates and may be used for non-integrable systems, just as well as for the integrable or superintegrable ones. The reduced phase space gets a new non-canonical bracket from the original phase space under this reduction since the invariance of the equations of motion is desired. Momentum map is a reduction procedure corresponding to a particular conserved quantity. Consider, for example, the system for the free rigid body. It is well known that the magnitude of the angular momentum vector is a constant of motion for this system. The momentum map, consisting of the angular momentum is given by

$$\begin{aligned}
L_1 &= \frac{P_\theta \sin\theta \cos\psi + P_\varphi \sin\psi - P_\psi \cos\theta \sin\psi}{\sin\theta}, \\
L_2 &= \frac{P_\varphi \cos\psi - P_\theta \sin\theta \sin\psi - P_\psi \cos\theta \cos\psi}{\sin\theta}, \\
L_3 &= P_\psi.
\end{aligned} \tag{20}$$

It is clear that this map reduces the six-dimensional phase space to the three dimensional one with the non-canonical coordinates $L_1, L_2, L_3$. The time evolution of the momentum map follows the Euler equations:

$$\dot{L}_1 = L_2\frac{L_3}{I_3} - L_3\frac{L_2}{I_2}, \quad \dot{L}_2 = L_3\frac{L_1}{I_1} - L_1\frac{L_3}{I_3}, \quad \dot{L}_3 = L_1\frac{L_2}{I_2} - L_2\frac{L_1}{I_1}, \tag{21}$$

where $I_j$, $(j = 1, 2, 3)$ are the inertia momenta with respect to the principal axes. The map (20) defines a Nambu structure with the 3-form $\omega^{(3)} = dL_1 \wedge dL_2 \wedge dL_3$ such that

$$\dot{L}_j = \omega^{(3)}(\widetilde{dL}_j, \widetilde{dH}_1, \widetilde{dH}_2), \tag{22}$$

where $H_1 = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2)$ is the reduced phase space, *i.e.*, the invariant sphere, and $H_2 = \frac{1}{2}\left(L_1^2/I_1 + L_2^2/I_2 + L_3^2/I_3\right)$ is the Hamiltonian for the reduced system. The trajectory $L(t)$ on the reduced phase space is the intersection of an ellipsoid and a sphere. The reduction operation mentioned above can be summarized in a differential geometric point of view as the following. (From now on, for the sake of clarity it will be concentrated on three-dimensional phase spaces. It is of course possible to extend it to higher dimensions using similar arguments).

Given a symplectic structure described by the 2-form in (3) and the Hamiltonian $H_0 = H_0(q_i, p_i)$, the transition to the Nambu structure is determined by the transformation

$$x_j = x_j(q_i, p_i), \tag{23}$$

comprising the components of a momentum map. This transformation must obey the conservation condition

$$\dot{x}_j = \omega^{(2)}(^\sharp dx_j, ^\sharp dH_0) = \omega^{(3)}(\widetilde{dx_j}, \widetilde{dH_1}, \widetilde{dH_2}). \tag{24}$$

If the reduced Hamiltonian is chosen as $H_1 = H_1(x_1, x_2, x_3)$, then the condition (24) shows that the transition functions satisfy the commutation relations

$$\{x_j, x_k\}_P = \epsilon_{jkl}\partial_{x_l}H_2, \quad , \quad (j, k, l = 1, 2, 3), \tag{25}$$

determining what the second Hamiltonian $H_2$ must be. In the rigid-body example stated above, the commutation relations (25) correspond to the angular momentum algebra and the calculation of the other Hamiltonian $\frac{1}{2}L^2$ is straightforward.

Clearly for the transformations $\dot{x}_j = 0$, such as the Hopf fibration (that will be considered later), the second Hamiltonian $H_2$ is of the form $H_2 = H_2(H_1)$ ,*i.e.*, any function of $H_1$. On the other hand the action of the momentum map on the symplectic 2-form $\omega^{(2)}$ can be represented by an operation such as $\hat{\mathcal{D}}\omega^{(2)} = \omega^{(3)}$ defined by

$$\hat{\mathcal{D}}\omega^{(2)} = \imath_{X_{x_1}}\omega^{(2)} \wedge \imath_{X_{x_2}}\omega^{(2)} \wedge \imath_{X_{x_3}}\omega^{(2)} = dx_1 \wedge dx_2 \wedge dx_3, \tag{26}$$

where $\imath$ is the usual interior product. For the 2-form $\omega^{(2)} = dq \wedge dp$ it is obvious that $\hat{\mathcal{D}}\omega^{(2)} = 0$. The invariance under the canonical transformations is the final remark about the momentum map. If $y_1, y_2, y_3$ are the transformed coordinates with respect to the new canonical coordinates then $dx_1 \wedge dx_2 \wedge dx_3 = dy_1 \wedge dy_2 \wedge dy_3$. This fact will be the main idea throughout the text.

# 3   Hopf Fibration and Behavior of Action-Angle Variables

In this section, Hopf fibration is given as model example for the momentum map and it is shown how to transform the action-angle variables under this reduction process.

It is well known that, in a two-dimensional phase space, the most suitable choice as canonical transformation for the action-angle variables is the set of polar coordinates $(r, \theta)$, just as the spherical coordinates for four-dimensional phase space shown in the following. Thus the question arises naturally: Is the set of spherical coordinates a suitable choice for the action-angle variables for a three-dimensional phase space? This section shows that the answer to this question is affirmative. The question may seem a bit trivial but it is meaningful when considering the bracket of the system, because such a choice must not effect the structure of the Nambu bracket. Next section confirms this fact.

Consider the 4-dimensional harmonic oscillator with the Hamiltonian

$$H_0 = \frac{1}{2}(p_1{}^2 + p_2{}^2 + q_1{}^2 + q_2{}^2), \tag{27}$$

describing a 3-sphere $S^3$. The Hopf fibration is given by the momentum-map components

$$x_1 = 2(q_1 q_2 + p_1 p_2), \quad x_2 = 2(q_2 p_1 - q_1 p_2), \quad x_3 = q_1{}^2 + p_1{}^2 - q_2{}^2 - p_2{}^2, \tag{28}$$

obeying $\dot{x}_j = 0$. Such a map converts the three-sphere $S^3$ into the two-sphere $S^2$, identified by the reduced Hamiltonian

$$H_1 = \frac{1}{2}(x_1{}^2 + x_2{}^2 + x_3{}^2)^{1/2} \tag{29}$$

Though a different set of spherical coordinates can be used for a four-dimensional sphere to define a canonical transformation, we prefer the following one used for the symmetrical top,

$$q_1 = r^{1/2} \cos(\frac{\theta}{2}) \cos[\frac{1}{2}(\alpha + \beta)], \qquad p_1 = r^{1/2} \cos(\frac{\theta}{2}) \sin[\frac{1}{2}(\alpha + \beta)], \tag{30}$$

$$q_2 = r^{1/2} \sin(\frac{\theta}{2}) \cos[\frac{1}{2}(\alpha - \beta)], \qquad p_2 = r^{1/2} \sin(\frac{\theta}{2}) \sin[\frac{1}{2}(\alpha - \beta)], \tag{31}$$

where $r = p_1{}^2 + p_2{}^2 + q_1{}^2 + q_2{}^2$ and $\alpha \in [0, 2\pi)$, $\theta \in [0, \pi)$, $\beta \in [0, 4\pi)$. Now if we let

$$r_1 = r^{1/2} \cos(\frac{\theta}{2}), \quad \theta_1 = \frac{1}{2}(\alpha + \beta), \quad r_2 = r^{1/2} \sin(\frac{\theta}{2}), \quad \theta_2 = \frac{1}{2}(\alpha - \beta), \tag{32}$$

we get the closed orbits in the $(q_i, p_i)$-planes:

$$q_1 = r_1 \cos \theta_1, \quad p_1 = r_1 \sin \theta_1, \quad q_2 = r_2 \cos \theta_2, \quad p_2 = r_2 \sin \theta_2, \tag{33}$$

where $r_1{}^2 + r_2{}^2 = r$. Consequently, the symplectic two-form $\omega^{(2)} = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ transforms to

$$\omega^{(2)} = r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2. \tag{34}$$

Obviously, this transformation describes the invariant action-angle variables

$$\omega^{(2)} = d\mathcal{I}_1 \wedge d\theta_1 + d\mathcal{I}_2 \wedge d\theta_2 \tag{35}$$

with the action variables $\mathcal{I}_1 = r_1{}^2/2$, $\mathcal{I}_2 = r_2{}^2/2$.

On the other hand, the transition to the Nambu 3-form $\omega^{(3)}$ is possible via the transition function $F = -8H_0{}^2$ such as

$$\omega^{(3)} = dF \wedge \omega^{(2)}, \tag{36}$$

or in an explicit form

$$dx_1 \wedge dx_2 \wedge dx_3 = 8r_1 r_2(r_1{}^2 + r_2{}^2) dr_1 \wedge dr_2 \wedge d(\theta_1 - \theta_2). \tag{37}$$

When written in terms of the action-angle variables, (37) follows

$$dx_1 \wedge dx_2 \wedge dx_3 = 16(\mathcal{I}_1 + \mathcal{I}_2) d\mathcal{I}_1 \wedge d\mathcal{I}_2 \wedge d(\theta_1 - \theta_2), \tag{38}$$

which has the more compact form

$$\omega^{(3)} = dG \wedge (d\mathcal{I}_i \wedge d\theta_i), \tag{39}$$

where $G = -8(\mathcal{I}_1 + \mathcal{I}_2)^2$. If one defines $\varphi = \theta_1 - \theta_2$ in (37), then

$$dx_1 \wedge dx_2 \wedge dx_3 = r^2 \sin \theta dr \wedge d\theta \wedge d\varphi. \tag{40}$$

At this stage it is not difficult to define the new action-angle variables. Indeed the rearrangement of (40) as the following

$$\omega^{(3)} = d(r^3/3) \wedge d(-\cos\theta) \wedge d\varphi = d\mathcal{J} \wedge d\mu \wedge d\varphi, \tag{41}$$

gives the action variable in terms of the volume $V$ enclosed in the phase space;

$$\mathcal{J} = \frac{1}{4\pi}V = \frac{1}{8\pi}\ \epsilon_{klm}\int x_k dx_l \wedge dx_m, \qquad k,l,m = 1,2,3. \tag{42}$$

Note that $\mathcal{J}$ does not have the dimensions of an angular momentum. Therefore we should keep this in mind when we talk about "action". Perhaps the best way to emphasize the action-angle variables is to say "area-angle" variables for the symplectic structures and "volume-solid angle" variables for the Nambu structures. Indeed, if we consider the solid angle element $d\Omega = \sin\theta d\theta \wedge d\varphi$, the Nambu three-form can be expressed as

$$\omega^{(3)} = d\mathcal{J} \wedge d\Omega. \tag{43}$$

# 4   Free Symmetrical Top

In this final section, the illustration of the argument is given by the example of free symmetrical top. We show that the Nambu bracket gives the correct equations of motion when expressed in the new variables $\mathcal{J}, \mu, \varphi$ in (41). We shall seek a set of equations of motion similar to (12),

$$\dot{\mathcal{J}} = \{\mathcal{J}, K_1, K_2\}_N = 0, \quad \dot{\mu} = \{\mu, K_1, K_2\}_N, \quad \dot{\varphi} = \{\varphi, K_1, K_2\}_N, \tag{44}$$

where $K_1 = K_1(\mathcal{J}, \mu, \phi)$ and $K_2 = K_2(\mathcal{J}, \mu, \phi)$ are the transformed forms of $H_1$ and $H_2$ respectively. The first equation is a direct consequence of the Liouville theorem corresponding to the invariance of the volume in the phase space.

A free symmetrical top is a free rigid body with $I_1 = I_2$. Thus the Euler equations of motion (21) reduce to

$$\dot{L}_1 = L_3\left(\frac{1}{I_3} - \frac{1}{I_1}\right)L_2 = \omega L_2, \qquad \dot{L}_2 = -\omega L_1, \qquad \dot{L}_3 = 0. \tag{45}$$

The solution of these equations states that the vector $\mathbf{L}$ rotates uniformly about the vertical axis of the body, say $z$, with the constant frequency $\omega$. Also the angle $\theta$ between the $z$-axis and the vector $\mathbf{L}$ is constant. This clearly implies $\dot{\mu} = 0$.

Since the phase space of the rigid body is identified with the angular-momentum sphere, it is easy to construct the action variable such as $\mathcal{J} = (2H_2)^{3/2}/3$, which induces

$$K_1 = \frac{(3\mathcal{J})^{2/3}}{2}\left[\frac{1}{I_1} - \mu^2\left(\frac{1}{I_1} - \frac{1}{I_3}\right)\right], \tag{46}$$

$$K_2 = (3\mathcal{J})^{2/3}/2. \tag{47}$$

We are now in a position to perform the brackets in (44). If we write the brackets explicitly, it is clear that

$$\dot{\mathcal{J}} = \partial_\mu K_1 \partial_\varphi K_2 - \partial_\varphi K_1 \partial_\mu K_2 = 0, \tag{48}$$

$$\dot{\mu} = \partial_\varphi K_1 \partial_{\mathcal{J}} K_2 - \partial_{\mathcal{J}} K_1 \partial_\varphi K_2 = 0, \tag{49}$$

$$\dot{\varphi} = \partial_{\mathcal{J}} K_1 \partial_\mu K_2 - \partial_\mu K_1 \partial_{\mathcal{J}} K_2 = L_3\left(\frac{1}{I_3} - \frac{1}{I_1}\right). \tag{50}$$

Thus the immediate solution of (50) is $\varphi = \omega t +$ constant. This is in accordance with the well known results about the symmetrical top.

# Acknowledgments

# References

[1] Y. Nambu: Phys.Rev. D **7** (1973) 2405.

[2] F. Bayen and M. Flato: Phys. Rev. D **11** (1975) 3049.

[3] N. Mukunda and E. Sudarshan: Phys. Rev. D **13** (1976) 2846.

[4] L. Takhtajan: Comm. Math. Phys. **160** (1994) 295.

[5] R. Chatterjee: Lett. Math. Phys. **36** (1996) 117.

[6] T. L. Curtright and C. K. Zachos: Phys. Rev. D **68** (2003) 085001, **hep-th/0212267**.

[7] A. Teğmen and A. Verçin: Int. J. Mod. Phys. A **19**(3) (2004) 393, **math-ph/0212070**.

[8] J. Marsden and T.S. Ratiu: *Introduction to Mechanics and Symmetry* Springer-Verlag 1994.